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Propagation of singularities in the Cauchy problem for a class of degenerate hyperbolic operators

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Abstract

We consider second order degenerate hyperbolic Cauchy problems, the degeneracy coming either from low regularity (less than Lipschitz continuity) of the coefficients with respect to time, or from weak hyperbolicity. In the weakly hyperbolic case, we assume an intermediate condition between effective hyperbolicity and the Levi condition. We construct the fundamental solution and study the propagation of singularities using an unified approach to these different kinds of degeneracy.

Keywords: Degenerate hyperbolic Cauchy problems; Fundamental solution; Propagation of singularities

1. Introduction

We deal with the propagation of the singularities in the Cauchy problem

$$\begin{cases} P(t,x,D_t,D_x)u(t,x)=0, \\ u(0,x)=u_0(x), \\ \partial_t u(0,x)=u_1(x), \end{cases} \tag{1.1}$$

$(t,x)\in[0,T]\times\mathbf{R}^n$, for a second order hyperbolic operator

$$P=D_t^2-2Q_1(t,x,D_x)D_t-Q_2(t,x,D_x),\qquad D=\frac{1}{\sqrt{-1}}\partial,\ \ Q_j(t,x,\xi)\in C\big([0,T];S^j\big),\ \ j=1,2. \tag{1.2}$$

Here S^j denotes the space of all symbols $p(x,\xi)$ of order j in $\mathbf{R}^n$ satisfying

$$\sup_{x,\xi\in\mathbf{R}^n}\langle\xi\rangle^{-j+|\alpha|}\big|\partial_x^\beta\partial_\xi^\alpha p(x,\xi)\big|<+\infty,\qquad \langle\xi\rangle=\sqrt{1+|\xi|^2}.$$

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The hyperbolicity condition is expressed by

$$\big(Q_{1,p}(t,x,\xi)\big)^2 + Q_{2,p}(t,x,\xi) \geqslant 0, \tag{1.3}$$

$Q_{j,p}$ the principal symbol of Q_j, $j = 1, 2$. We prove results for general pseudodifferential operators in the strictly hyperbolic case

$$\big(Q_{1,p}(t,x,\xi)\big)^2 + Q_{2,p}(t,x,\xi) \geqslant \lambda_0|\xi|^2, \quad \lambda_0 > 0, \tag{1.4}$$

assuming that the principal symbols $Q_{1,p}$, $Q_{2,p}$ are positively homogeneous in the variable ξ, so that the flux of the bicharacteristic curves of P corresponds to canonical transformations in the cotangent bundle of $\mathbf{R}^n$.

When $Q_{1,p}^2 + Q_{2,p}$ vanishes at some point, the weakly hyperbolic case, we consider differential operators, that is we assume that Q_1, Q_2 are polynomials in the variable ξ.

We say that problem (1.1) is well-posed in the space X of functions or distributions in $\mathbf{R}^n$ if for every $u_0, u_1 \in X$ there is a unique solution $u \in C^1([0,T]; X)$. A result of well-posedness motivates the study of the propagation of singularities of the solution since these aspects of the Cauchy problem are deeply connected.

When P is strictly hyperbolic, it is well known that, if the symbols $Q_{j,p}$ are Lipschitz continuous in the variable t, then the Cauchy problem is well-posed in the Sobolev spaces $H^\infty = \bigcap_\mu H^\mu$ and $H^{-\infty} = \bigcup_\mu H^\mu$. When P is a differential operator, the well-posedness in C^∞ and in the space of distributions $\mathcal{D}'$ follows from the finite speed of propagation of the supports.

This may fail to be true either for a strictly hyperbolic operator with less than Lipschitz regularity in the time variable or for a weakly hyperbolic operator, even if now one takes $Q_{j,p} \in C^\infty([0,T]; S^j)$.

The research of classes of operators for which well-posedness still holds developed in these two directions.

In the strictly hyperbolic case, interesting results have been obtained in weakening the Lipschitz regularity with respect to the t variable. To this purpose, we recall that a function $f : I \to \mathbf{R}$, I a real interval, is said to be Log-Lipschitz continuous if it satisfies

$$\|f\|_{\mathrm{LL}(I)} := \sup_{\substack{t,s\in I \\ 0<|t-s|<1/2}} \frac{|f(t)-f(s)|}{|t-s||\log|t-s||} < +\infty.$$

From [7] and [12] (see also [1] for the case $Q_1 \neq 0$ and higher order operators), we know that under the Log-Lipschitz regularity

$$Q_{j,p} \in \mathrm{LL}\big([0,T]; S^j\big) \tag{1.5}$$

the Cauchy problem (1.1) is $H^{\pm\infty}$ well-posed and this condition is sharp as far as the modulus of continuity is concerned.

Another way to weaken the Lipschitz regularity, namely the singular behaviour of the first derivative

$$\big|f'(t)\big| \leqslant \frac{C}{|t-t_0|^q}, \quad C > 0,\ q \geqslant 1,$$

as t tends to a point $t_0 \in [0,T]$, say $t_0 = 0$, has been introduced in [8] and then studied in [3,9,14,15,17,18]. Now we know that the condition

$$t\partial_t Q_{j,p} \in B^0\big(]0,T]; S^j\big), \quad j = 1, 2, \tag{1.6}$$

ensures $H^{\pm\infty}$ well-posedness and that the exponent $q = 1$ is sharp as far as the powers t^q are concerned.

Coming to weakly hyperbolic operators with smooth coefficients in all variables, one can assume $Q_1 \equiv 0$ in (1.2) without any loss of generality, at least in microlocal analysis. Let us consider a differential operator $P = D_t^2 - Q_2(t,x,D_x)$,

$$Q_2(t,x,D_x) = \sum_{i,j=1}^n \alpha_{ij}(t,x) D_{x_i} D_{x_j} + \sum_{j=1}^n b_j(t,x) D_{x_j} + c(t,x),$$

with coefficients $\alpha_{ij}, b_j, c \in B^\infty([0,T] \times \mathbf{R}^n)$ and let us denote

$$p(t,x,\xi)=|\xi|^{-2}Q_{2,p}(t,x,\xi)=|\xi|^{-2}\sum_{i,j=1}^{n}\alpha_{ij}(t,x)\xi_i\xi_j,$$

$$s(t,x,\xi)=|\xi|^{-1}\sum_{j=1}^{n}b_j(t,x)\xi_j.$$

The Cauchy problem for P with arbitrary lower order terms is C^∞ well-posed if and only if the principal part $D_t^2-Q_{2,p}(t,x,D_x)$ is effectively hyperbolic. If this is not the case, then one needs to impose Levi conditions on $s(t,x,\xi)$. The problem of determining necessary and sufficient Levi conditions has not been completely solved, even if many deep results have been obtained. For analytic coefficients depending only on the time variable, the inequality

$$s(t,\xi)\leqslant C\sqrt{p(t,\xi)},\quad C>0, \tag{1.7}$$

is a sufficient Levi condition [11]. This holds true for general analytic coefficients in one dimension of space [20].

After having observed that for C^∞ coefficients depending only on the t variable, the effective hyperbolicity is expressed by

$$\partial_t^2 p(t,\xi)>0$$

at all points (t,ξ) where $p(t,\xi)=\partial_t p(t,\xi)=0$, that is by

$$\sum_{j=0}^{2}\left|\partial_t^j p(t,\xi)\right|\neq 0,$$

an intermediate assumption between such a condition and the Levi condition (1.7) has been proposed in [10], namely

$$\sum_{j=0}^{k}\left|\partial_t^j p(t,\xi)\right|\neq 0,\qquad s(t,\xi)\leqslant C\big(p(t,\xi)\big)^\gamma,\quad C>0, \tag{1.8}$$

for an integer $k\geqslant 2$ and an exponent $\gamma\in[0,1/2]$. There the authors prove $H^{\pm\infty}$ well-posedness under this assumption with $\gamma\geqslant 1/2-1/k$ for C^∞ coefficients. Such a relation between γ and k is sharp since the Cauchy problem for

$$P=D_t^2-t^{2\ell}D_x^2+t^\nu D_x$$

is well-posed in C^∞ if and only if $\nu\geqslant\ell-1$ [16]. One observes also that for $k=2$ (effective hyperbolicity) we can take $\gamma=0$, that is no Levi condition is necessary. On the other hand, for $k=+\infty$ we have to take $\gamma=1/2$ that is we have to impose the Levi condition (1.7).

Lower order terms have been allowed to depend also on the space variables in [13] under the stronger condition (for $k>2$) $\gamma\geqslant 1/2-1/2(k-1)$.

In the paper [2] we considered weakly hyperbolic operators of the type

$$\begin{cases} P=D_t^2-\alpha(t)Q(t,x,D_x)+b(t,x,D_x)+c(t,x),\\ Q(t,x,D_x)=\displaystyle\sum_{i,j=1}^{n}a_{ij}(t,x)D_{x_i}D_{x_j},\\ b(t,x,D_x)=\displaystyle\sum_{j=1}^{n}b_j(t,x)D_{x_j},\\ \alpha(t)\geqslant 0,\quad Q(t,x,\xi)\geqslant\lambda_0|\xi|^2,\ \lambda_0>0, \end{cases} \tag{1.9}$$

with coefficients

$$\alpha\in C^\infty\big([0,T]\big),\qquad a_{ij},b_j,c\in C^\infty\big([0,T]\times\mathbf{R}^n\big). \tag{1.10}$$

Only the common factor $\alpha(t)$ may vanish in the quadratic form $\alpha(t)Q(t,x,\xi)$ and it has only zeroes of finite order less or equal to k. In particular $\alpha(t)$ can be zero only at a finite number of isolated points in $[0,T]$.

To be precise, in [2] we took coefficients $a_{ij} = a_{ij}(x)$ in Q but our results there hold true in the slightly more general case $a_{ij} = a_{ij}(t,x)$ (see Section 2 here).

We proved $H^{\pm\infty}$ well-posedness assuming that there is $k \geqslant 2$ such that

$$\sum_{h=0}^{k} \left|\alpha^{(h)}(t)\right| \neq 0, \qquad \left|\partial_x^\beta b_j(t,x)\right| \leqslant C_\beta\bigl(\alpha(t)\bigr)^\gamma, \quad \gamma \geqslant \frac{1}{2} - \frac{1}{k}, \tag{1.11}$$

so recovering, for this particular class of operators, the sharp relation between the order of zero k and the Levi exponent γ.

This contains and improves the results of [13] only in the case of dimension of space $n = 1$ since we cannot consider general quadratic forms $\sum_{i,j=1}^{n} \alpha_{ij}\xi_i\xi_j \geqslant 0$, $n > 1$, even with coefficients α_{ij} depending only on t.

In the paper [2], we have also shown that the results of $H^{\pm\infty}$ well-posedness for strictly hyperbolic operators satisfying either condition (1.5) or condition (1.6) and for weakly hyperbolic operators of the type (1.9) fulfilling assumption (1.11) can be obtained by the same method. In this sense, we have found that these weakly and strictly hyperbolic operators belong to the same class of degenerate hyperbolic operators.

This common method consisted of the following steps:

(1) Factorization of the principal part of P by means of regularized characteristic roots. This gives a factorization of the full operator P with a remainder.

(2) Given a function $f = f(t,x)$, reduction of the scalar equation $Pu = f$ to an equivalent 2×2 system

$$LU = F, \quad L = \partial_t - i\Lambda(t,x,D_x) + A(t,x,D_x). \tag{1.12}$$

Here $\Lambda(t,x,\xi)$ is a real diagonal matrix of symbols of order 1 whose entries coincide with the characteristic roots of P for large ξ.

The matrix $A(t,x,\xi)$ comes from the remainder in the factorization of the full operator P. $A(t)$ is still of positive order (less or equal to 1) at each fixed time t but it satisfies

$$\int_0^T \bigl|A(s,x,\xi)\bigr|\,ds \leqslant c_0 + \delta\log\langle\xi\rangle. \tag{1.13}$$

(3) A priori energy estimates in Sobolev spaces for the operator L (with a δ-loss of derivatives).

Another common approach, based on the Littlewood–Paley decomposition, can be found in [6].

Here, our aim is to construct the fundamental solution for such an operator L in order to investigate the propagation of the singularities with respect to the space variable. This problem was only partially considered in [2] for some special strictly hyperbolic operators. By means of the fundamental solution, we also re-obtain the $H^{\pm\infty}$ well-posedness of the Cauchy problem with a δ-loss of derivatives, so making here not necessary the step (3) above.

The fundamental solution that we construct, is a continuous family of bounded operators $E(t,s)$ in $H^{-\infty}$, $t,s \in [0,T]$ such that

$$\begin{cases} LE(t,s) = 0, \\ E(s,s) = I. \end{cases}$$

The unique solution $U \in C([0,T]; H^\mu)$ of the Cauchy problem $LU(t,x) = F(t,x)$, $U(0,x) = U_0$, is then given by Duhamel's formula $U(t) = E(t,0)U_0 + \int_0^t E(t,s)F(s)\,ds$.

We use the method of multi-products of Fourier integral operators of [19], representing $E(t,s)$ as a series

$$E(t,s) = \sum_{\nu=0}^{\infty} p_{\nu,\Psi^\nu}(t,s,x,D_x)$$

of such Fourier operators with amplitude $p_\nu(t,s,x,\xi)$ and phase-function Ψ^ν.

The bound (1.13) leads to amplitudes such that

$$\bigl|p_\nu(t,s,x,\xi)\bigr| \leqslant \frac{1}{\nu!}\bigl(\delta\log\langle\xi\rangle\bigr)^\nu.$$

In particular, $E(t,s)$ is a continuous operator from $H^{s+\delta}$ to H^s and this gives the $H^{\pm\infty}$ well-posedness with a δ-loss of derivatives.

Each phase Ψ^ν is the generating function of the composition of ν bicharacteristic curves. So, micro-local singularities (with respect to the space variable) of the solution $u(t,\cdot)$ in (1.1) at the time t, can be found in points of composed bicharacteristic curves that start from the wave front sets of the Cauchy data and that may change bicharacteristic curve at times $t_1,\dots,t_\nu$, $t \geqslant t_\nu \geqslant \dots \geqslant t_1 \geqslant 0$.

In the strictly hyperbolic case, we show that this may happen only if the coefficients of P are singular at each t_k, $k=1,\dots,\nu$, with respect to time.

Concerning weakly hyperbolic operators with smooth coefficients, we show that this is possible only if the characteristics coincide, that is if the coefficient $\alpha(t)$ in (1.9) vanishes, at each time t_k.

While it is well known that such a behaviour of the singularities really appears in weakly hyperbolic problems, this flux of composed bicharacteristics may be a little bit unexpected in the strictly hyperbolic case. One could think that the regularity in x of the solution is not influenced by the singularity in time of the coefficients. We refer to [4] and [5] for simple examples where such a type of propagation really takes place in the strictly hyperbolic Cauchy problem with non regular coefficients in the time variable.

In order to keep this paper as self contained as possible, in Section 2 we briefly recall the first two steps of our method, that is the reduction of the scalar Cauchy problem (1.1) to the Cauchy problem for a system L satisfying (1.12), (1.13). In Section 3 we construct the fundamental solution for the operator L. In Section 4, we investigate the propagation of singularities by means of the fundamental solution.

2. The reduction to a first order system

The aim of this section is to show that the Cauchy problem (1.1) can be reduced to a first order system $LU=F$, $U(0)=U_0$, with $L=\partial_t - i\Lambda + A$ as in (1.12) and with (1.13) specified by the following estimate of the symbol A:

$$\begin{cases} A\in L^1\big([0,T];S^1\big),\\ \big|\partial_\xi^\alpha\partial_x^\beta A(t,x,\xi)\big| \leqslant \rho_{\alpha\beta}(t,\xi)\langle\xi\rangle^{-|\alpha|},\\ \rho_{\alpha\beta}\in C\big([0,T]\times\mathbf{R}^n\big),\\ \displaystyle\int_0^T \rho_{\alpha\beta}(t,\xi)\,dt \leqslant \delta_{\alpha\beta}\log\big(1+\langle\xi\rangle\big), \quad \delta_{\alpha\beta}>0. \end{cases} \tag{2.1}$$

2.1. Strictly hyperbolic operators

Let us consider the operator (1.2) under the strict hyperbolicity assumption (1.4) and satisfying either condition (1.5) or condition (1.6).

(1) Factorization.

Let us denote $\tau=\lambda_j(t,x,\xi)$, $j=1,2$, the roots of

$$\tau^2 - 2Q_{1,p}(t,x,\xi)\tau - Q_{2,p}(t,x,\xi) = 0$$

and let us introduce the mollified symbols

$$\tilde\lambda_j(t,x,\xi) = \int \lambda_j(\tau,x,\xi)\varrho\big((t-\tau)\langle\xi\rangle\big)\langle\xi\rangle\,d\tau,$$

with $\varrho\in C_0^\infty(\mathbf{R})$, $0\leqslant\varrho\leqslant 1$, $\int\varrho(\tau)\,d\tau=1$, $\lambda_j(\tau,x,\xi)=\lambda_j(T,x,\xi)$ for $\tau\geqslant T$, $\lambda_j(\tau,x,\xi)=\lambda_j(0,x,\xi)$ for $\tau\leqslant 0$.

The symbols

$$A_j^{(0)} = \lambda_j - \tilde\lambda_j, \qquad A_j^{(1)} = \langle\xi\rangle^{-1}\partial_t\tilde\lambda_j$$

fulfill

$$\int_0^T \big|\partial_x^\beta\partial_\xi^\alpha A_j^{(k)}(t,x,\xi)\big|\,dt \leqslant \delta_{\alpha\beta}\langle\xi\rangle^{-|\alpha|}\log\big(1+\langle\xi\rangle\big).$$

In fact, under the assumption (1.5), we have the more strict estimate

$$\left|\partial_x^\beta \partial_\xi^\alpha A_j^{(k)}(t,x,\xi)\right| \leqslant C_{\alpha\beta}\langle\xi\rangle^{-|\alpha|}\log\bigl(1+\langle\xi\rangle\bigr).$$

On the other hand, if the principal symbol of P satisfies the condition (1.6), then we have both

$$A_j^{(k)} \in C\bigl([0,T];S^1\bigr)$$

and

$$tA_j^{(k)} \in C\bigl([0,T];S^0\bigr),$$

thus we have

$$\int_0^T \left|\partial_x^\beta \partial_\xi^\alpha A_j^{(k)}(t,x,\xi)\right| dt \leqslant C_{\alpha\beta}\int_0^{2\langle\xi\rangle^{-1}} \langle\xi\rangle^{1-|\alpha|}\,dt + C_{\alpha\beta}\int_{2\langle\xi\rangle^{-1}}^T \frac{1}{t}\langle\xi\rangle^{-|\alpha|}\,dt \leqslant \delta_{\alpha\beta}\langle\xi\rangle^{-|\alpha|}\log\bigl(1+\langle\xi\rangle\bigr).$$

So, we can factorize P as follows

$$P(t,x,D_t,D_x) = \bigl(D_t - \tilde\lambda_2(t,x,D_x)\bigr)\bigl(D_t - \tilde\lambda_1(t,x,D_x)\bigr) + R_0(t,x,D_x)\langle D_x\rangle + R_1(t,x,D_x)D_t, \tag{2.2}$$

with $R_j \in C([0,T];S^1)$ satisfying

$$\int_0^T \left|\partial_x^\beta \partial_\xi^\alpha R_j(t,x,\xi)\right| dt \leqslant \delta_{\alpha\beta}\langle\xi\rangle^{-|\alpha|}\log\bigl(1+\langle\xi\rangle\bigr). \tag{2.3}$$

(2) Reduction to system.

For a given scalar function $u(t,x)$, let us define the vector $V=(v_0,v_1)$ by

$$\begin{cases} v_0 = \langle D_x\rangle u, \\ v_1 = \bigl(D_t - \tilde\lambda_1(t,x,D_x)\bigr)u. \end{cases} \tag{2.4}$$

From the factorization (2.2), after a straightforward diagonalization, there is an elliptic symbol $M \in C([0,T];S^0)$ such that the Cauchy problem (1.1) in the unknown u is equivalent to the Cauchy problem in the unknown $U = M(t,x,D_x)V$

$$\begin{cases} LU = 0, \\ U(0,x) = U_0, \end{cases} \tag{2.5}$$

with the operator L given by

$$L = \partial_t - \begin{pmatrix} i\tilde\lambda_1(t,x,D_x) & 0 \\ 0 & i\tilde\lambda_2(t,x,D_x) \end{pmatrix} + A(t,x,D_x)$$

and where the 2×2 matrix $A(t,x,\xi) \in C([0,T];S^1)$ satisfies (2.1) thanks to (2.3).

2.2. Weakly hyperbolic operators

Now let us consider an operator P given by (1.9), (1.10) and satisfying condition (1.11).

(1) Factorization.

Let us define

$$\omega(t,\xi) = \sqrt{1+\alpha(t)\langle\xi\rangle^2}$$

and let us approximate the characteristics $\pm\lambda = \sqrt{\alpha(t)Q(t,x,\xi)}$ by

$$\tilde\lambda(t,x,\xi) = \sqrt{\alpha(t)+\langle\xi\rangle^{-2}}\sqrt{Q(t,x,\xi)}$$

so that

$$\tilde{\lambda}(t,x,\xi) = \langle\xi\rangle^{-1}\omega(t,\xi)\sqrt{Q(t,x,\xi)}.$$

Let us observe that

$$\omega \in C\big([0,T];S^1\big), \qquad \omega^{-1} \in C\big([0,T];S^0\big), \qquad \sqrt{\alpha}\omega^{-1} \in C\big([0,T];S^{-1}\big).$$

Furthermore, from (1.11), the symbols

$$\beta_0(t,\xi) := \alpha'(t)\langle\xi\rangle^2\omega^{-2}(t,\xi) = \frac{\alpha'(t)}{\alpha(t)+\langle\xi\rangle^{-2}} \tag{2.6}$$

and

$$\beta_1(t,x,\xi) := b(t,x\xi)\omega^{-1}(t,\xi) = \frac{b(t,x,\xi)}{\langle\xi\rangle\sqrt{\alpha(t)+\langle\xi\rangle^{-2}}} \tag{2.7}$$

can be taken as entries of a matrix A satisfying (2.1). In fact, for any positive integer N, Lemma 1 in [11] implies that the function $\alpha^{1/N}$ is absolutely continuous so we can write

$$\beta_0(t,\xi) = \frac{\alpha'(t)}{(\alpha(t)+\langle\xi\rangle^{-2})^{1-1/N}} \cdot \frac{1}{(\alpha(t)+\langle\xi\rangle^{-2})^{1/N}}$$

in order to get

$$\beta_0 \in L^1\big([0,T];S^{2/N}\big).$$

For β_1, taking $\gamma = 1/2 - 1/k$ in (1.11), we write

$$\beta_1(t,x,\xi) = \frac{b(t,x,\xi)}{\langle\xi\rangle(\alpha(t)+\langle\xi\rangle^{-2})^{\gamma}} \cdot \frac{1}{(\alpha(t)+\langle\xi\rangle^{-2})^{1/k}}$$

and we obtain

$$\beta_1 \in C\big([0,T];S^{2/k}\big)$$

since

$$\frac{b(t,x,\xi)}{\langle\xi\rangle(\alpha(t)+\langle\xi\rangle^{-2})^{\gamma}}$$

is of order zero by assumption. Also the last condition in (2.1) is satisfied by the entries β_0, β_1 of A. In order to check this, one uses that $\alpha(t)$ has only zeros of finite order less or equal to k. In particular they are a finite number of isolated points in $[0,T]$. In a neighborhood of such a zero one just takes into account that

$$\int_0^T \frac{1}{(t^k+\langle\xi\rangle^{-2})^{1/k}}\,dt \leqslant \int_0^{\langle\xi\rangle^{-2/k}} \frac{1}{\langle\xi\rangle^{-2/k}}\,dt + \int_{\langle\xi\rangle^{-2/k}}^T \frac{1}{t}\,dt = 1 + \log\frac{T}{\langle\xi\rangle^{-2/k}}$$

and that also α' vanishes at that point changing sign from minus to plus (cf. Lemmas 1 and 2 in [10]).

So far, we obtain the following factorization of P

$$P(t,x,D_t,D_x) = \big(D_t+\tilde{\lambda}(t,x,D_x)\big)\big(D_t-\tilde{\lambda}(t,x,D_x)\big) + R(t,x,D_x)\omega(t,D_x) \tag{2.8}$$

with $R \in L^1([0,T];S^1)$ such that

$$R(t,x,\xi) = a(t,x,\xi)\beta_0(t,\xi) + b(t,x,\xi)\beta_1(t,x,\xi), \quad a,b \in C\big([0,T],S^0\big), \tag{2.9}$$

hence such that

$$\int_0^T \big|\partial_x^\beta\partial_\xi^\alpha R(t,x,\xi)\big|\,dt \leqslant \delta_{\alpha\beta}\langle\xi\rangle^{-|\alpha|}\log\big(1+\langle\xi\rangle\big). \tag{2.10}$$

(2) Reduction to system.

For a given scalar function $u(t,x)$, this time we define the vector $V=(v_0,v_1)$ by

$$\begin{cases} v_0=\omega(t,D_x)u, \\ v_1=\big(D_t-\tilde{\lambda}(t,x,D_x)\big)u. \end{cases} \tag{2.11}$$

From the factorization (2.8), also here there is an elliptic symbol $M\in C([0,T];S^0)$ such that the Cauchy problem (1.1) in the unknown u is equivalent to the Cauchy problem (2.5) in the unknown $U=M(t,x,D_x)V$ for an operator

$$L=\partial_t-\begin{pmatrix} i\tilde{\lambda}(t,x,D_x) & 0 \\ 0 & -i\tilde{\lambda}(t,x,D_x) \end{pmatrix}+A(t,x,D_x)$$

where, from (2.9), the matrix A has the structure

$$A(t,x,\xi)=A_0(t,x,\xi)\beta_0(t,\xi)+A_1(t,x,\xi)\beta_1(t,x,\xi),\quad A_0,A_1\in C\big([0,T],S^0\big),$$

β_0, β_1 defined by (2.6), (2.7), and it satisfies (2.1) thanks to (2.10).

3. The fundamental solution

Provided that T is sufficiently small, we construct the fundamental solution for the operator L as a continuous family of bounded operators $E(t,s)$ in $H^{-\infty}$, $t,s\in[0,T]$, satisfying

$$\begin{cases} LE(t,s)=0, \\ E(s,s)=I. \end{cases} \tag{3.1}$$

More precisely, there is a positive δ such that $E(t,s)$ is continuous from $H^{\mu+\delta}$ to H^{μ} for every μ. So, for any given Cauchy data $U_0\in H^{\mu+\delta}$, $F\in C([0,T];H^{\mu+\delta})$, the unique solution $U\in C([0,T];H^{\mu})$ of the Cauchy problem $LU(t,x)=F(t,x)$, $U(0,x)=U_0$, is given by Duhamel's formula

$$U(t)=E(t,0)U_0+\int_0^t E(t,s)F(s)\,ds. \tag{3.2}$$

We use the method of multi-products of Fourier integral operators by [19].

In the diagonal part of the symbol of L, we can put again the true roots λ_j of P since the difference $\tilde{\lambda}_j-\lambda_j$ can be taken as an entry of a matrix A satisfying (2.1).

The homogeneous symbols $\lambda_j(t,x,\xi)$ in the variable ξ, give canonical transformations $\mathcal{C}_j(t,s)$ in the cotangent bundle of $\mathbf{R}^n$

$$\mathcal{C}_j(t,s):(y,\eta)\mapsto\big(x^j,\xi^j\big).$$

They are defined by the Hamilton–Jacobi equations

$$\begin{cases} \dfrac{dx^j}{dt}=\nabla_\xi\lambda_j\big(t,x^j,\xi^j\big), \\ \dfrac{d\xi^j}{dt}=-\nabla_x\lambda_j\big(t,x^j,\xi^j\big), \\ \big(x^j,\xi^j\big)_{|t=s}=(y,\eta). \end{cases} \tag{3.3}$$

The generating phase-functions of the transformations $\mathcal{C}_j(t,s)$ are the solutions $\varphi_j=\varphi_j(t,s;x,\eta)$ of the eikonal equations

$$\begin{cases} \partial_t\varphi_j=-\lambda_j(t,x,\nabla_x\varphi_j), \\ \varphi_{j|_{t=s}}=x\cdot\eta, \end{cases} \tag{3.4}$$

since they satisfy

$$\begin{cases} y = \nabla_\eta \varphi_j\big(t, s; x^j, \eta\big), \\ \xi^j = \nabla_x \varphi_j\big(t, s; x^j, \eta\big). \end{cases} \tag{3.5}$$

The solution φ_j of (3.4) exists uniquely and satisfies also

$$\partial_s \varphi_j(t, s, x, \eta) = \lambda_j\big(s, \nabla_\xi \varphi_j(t, s, x, \eta), \eta\big), \quad t, s \in [0, T], \tag{3.6}$$

provided that T is sufficiently small.

We need also the multi-phase-functions

$$\Psi^\nu(t, t_1, \dots, t_\nu, s; x, \eta)$$

which are defined as the generating functions of the composed canonical transformations

$$\mathcal{C}^{(\nu)}(t, t_1, \dots, t_\nu, s) = \mathcal{C}_{j_1}(t, t_1)\mathcal{C}_{j_2}(t_1, t_2) \cdots \mathcal{C}_{j_\nu}(t_{\nu-1}, t_\nu)\mathcal{C}_{j_{\nu+1}}(t_\nu, s)$$

where each $\mathcal{C}_{j_k}$ is either $\mathcal{C}_1$ or $\mathcal{C}_2$. The points (X_ν^k, Ξ_ν^k) defined inductively for $k = 1, \dots, \nu$ by

$$\big(X_\nu^k, \Xi_\nu^k\big) = \mathcal{C}_{j_k}\big(t_k, t_{k-1}, X_\nu^{k-1}, \Xi_\nu^{k-1}\big), \qquad \big(X_\nu^0, \Xi_\nu^0\big) = \big(\nabla_\eta \Psi^\nu(t, t_1, \dots, t_\nu, s; x, \eta), \eta\big)$$

are called the critical points. The multi-phase-functions satisfy the eikonal equations

$$\begin{cases} \dfrac{d}{dt_k}\Psi^\nu(t, t_1, \dots, t_\nu, s; x, \eta) = \lambda_{j_k}\big(t_k, X_\nu^k, \Xi_\nu^k\big) - \lambda_{j_{k+1}}\big(t_k, X_\nu^k, \Xi_\nu^k\big), \\ \Psi^\nu(t, \dots, t_{k-1}, t_k, t_{k+1}, \dots, s)_{|_{t_k = t_{k+1}}} = \Psi^{\nu-1}(t, \dots, t_{k-1}, t_{k+1}, \dots, s) \end{cases} \tag{3.7}$$

for $k = 0, \dots, \nu + 1$, $t_0 = t$, $t_{\nu+1} = s$, $\lambda_{j_0} = \lambda_{j_{\nu+1}} = 0$, $\Psi^0 = \varphi_j$.

For $\varphi(x, \eta)$ a real homogeneous phase function of order 1 and an amplitude $a(x, \eta)$ of order m, we denote by $a_\varphi = a_\varphi(x, D_x)$ the Fourier integral operator from $H^{\mu+m}(\mathbf{R}^n)$ to $H^\mu(\mathbf{R}^n)$ given by

$$a_\varphi(x, D_x)v(x) = (2\pi)^{-n} \int e^{i\varphi(x,\eta)} a(x, \eta)\hat{v}(\eta)\, d\eta,$$

$\hat{v}$ the Fourier transform of v. We use also the notation $a = \sigma(a_\varphi)$.

From [19, Chapter 10, Theorem 6.8], for $p \in S^{m_1}$, $q \in S^{m_2}$, $t \geqslant t_1 \geqslant \cdots \geqslant t_{\nu_1} \geqslant t_{\nu_1+1} \geqslant \cdots \geqslant t_\nu$, we have

$$p_{\Psi^{\nu_1}(t, \dots, t_{\nu_1})} q_{\Psi^{\nu_2}(t_{\nu_1+1}, \dots, t_\nu)} = a_{\Psi^\nu(t, \dots, t_\nu)}, \quad \nu = \nu_1 + \nu_2, \tag{3.8}$$

with $a \in S^{m_1+m_2}$.

Now we are ready to construct the fundamental solution as the limit of a converging sequence of bounded operators in Sobolev spaces. Let us consider the operators

$$I_\varphi = \begin{pmatrix} I_{\varphi_1}(t, s) & 0 \\ 0 & I_{\varphi_2}(t, s) \end{pmatrix}, \qquad R_\varphi(t, s) = L I_\varphi(t, s), \tag{3.9}$$

and let us define the sequence

$$W_1(t, s) = -i R_\varphi(t, s), \qquad W_{\nu+1}(t, s) = \int_s^t W_1(t, \tau) W_\nu(\tau, s)\, d\tau, \quad \nu \geqslant 1. \tag{3.10}$$

Theorem 3.1. *Consider the operator*

$$L = \partial_t - i\Lambda(t, x, D_x) + A(t, x, D_x),$$

where Λ is a diagonal matrix of real symbols of order one and A satisfies (2.1)*. Consider moreover the sequence* (3.10) *defined by means of* (3.9)*. For a sufficiently small T, there exists $\delta > 0$ such that for every μ the series*

$$E(t, s) = I_\varphi(t, s) + \int_s^t I_\varphi(t, \tau) \sum_{\nu=1}^\infty W_\nu(\tau, s)\, d\tau \tag{3.11}$$

defines a continuous operator from $H^{\mu+\delta}$ to H^μ which satisfies (3.1).

Proof. If in (3.11) we have a well-defined bounded operator $E(t,s)$, then it easy to check that it satisfies (3.1), so we have only to prove that the series converges.

Let us consider the sequence

$$E_N(t,s) = I_\varphi(t,s) + \int_s^t I_\varphi(t,\tau) \sum_{\nu=1}^N W_\nu(\tau,s)\,d\tau.$$

From (3.8), the entries of the matrix

$$W_\nu(t,s) = \int_s^t \cdots \int_s^{t_{\nu-2}} W_1(t,t_1)\cdots W_1(t_{\nu-1},s)\,dt_{\nu-1}\cdots dt_1$$

are Fourier integral operators with the multi-products $\Psi^\nu(t,t_1,\dots,t_\nu,s;x,\eta)$ as phase-functions.

From (2.1) and the definition of W_1, we have

$$\begin{cases} \left|\partial_\xi^\alpha \partial_x^\beta \sigma\big(W_1(t,s)\big)\right| \leqslant \rho_{\alpha\beta}(t,s,\xi)\langle\xi\rangle^{-|\alpha|}, \\ \int_0^T \rho_{\alpha\beta}(t,s,\xi)\,dt \leqslant \delta_{\alpha\beta}\log\big(1+\langle\xi\rangle\big), \end{cases} \tag{3.12}$$

with positive symbols $\rho_{\alpha,\beta} \in L^1([0,T]^2; S^1)$. So, denoting

$$\rho_l(t,\xi) = \sup_{|\alpha+\beta|\leqslant l,\, s\in[0,T]} \rho_{\alpha,\beta}(t,s,\xi),$$

from [19, Chapter 10, formula (6.94)], for every $l \in \mathbf{Z}_+$ there is $l' \in \mathbf{Z}_+$ such that for $|\alpha+\beta| \leqslant l$, $t_0 = t$, we have

$$\left|\partial_\xi^\alpha \partial_x^\beta \sigma\big(W_1(t,t_1)W_1(t_1,t_2)\cdots W_1(t_{\nu-1},s)\big)\right| \leqslant c_l^{\nu-1}\langle\xi\rangle^{-|\alpha|}\prod_{j=0}^{\nu-1}\rho_{l'}(t_j,\xi).$$

By symmetry, taking (3.12) into account, one obtains

$$\int_s^t \cdots \int_s^{t_{\nu-1}} \prod_{j=1}^{\nu} \rho_{l'}(t_j,\xi)\,dt_\nu\cdots dt_1 \frac{1}{\nu!}\int_s^t \cdots \int_s^t \prod_{j=1}^{\nu}\rho_{l'}(t_j,\xi)\,dt_\nu\cdots dt_1 \leqslant \frac{1}{\nu!}\big(\delta_{l'}\log\big(1+\langle\xi\rangle\big)\big)^\nu$$

with $\delta_{l'} = \sup_{|\alpha+\beta|\leqslant l'}\delta_{\alpha,\beta}$. So, for $|\alpha+\beta| \leqslant l$, we get

$$\left|\partial_\xi^\alpha \partial_x^\beta \sigma(E_N)(t,s;x,\xi)\right| \leqslant C\langle\xi\rangle^{-|\alpha|}\sum_{\nu=0}^N \frac{(c_l\delta_{l'}\log(1+\langle\xi\rangle))^\nu}{\nu!} \leqslant C\langle\xi\rangle^{c_l\delta_{l'}-|\alpha|}.$$

Now, we can fix a positive integer l_0 and a positive constant $M > 0$ such that

$$\|p_{\Psi^\nu}u\|_0 \leqslant M|p|_{l_0}^{(m)}\|u\|_m, \quad |p|_{l_0}^{(m)} := \sup_{|\alpha|+|\beta|\leqslant l_0}\sup_{x,\xi}\langle\xi\rangle^{-m+|\alpha|}\left|\partial_\xi^\alpha\partial_x^\beta p(x,\xi)\right|,$$

for any $p \in S^m$ and every ν. If we take

$$\delta = c_{l_0}\delta_{l_0'},$$

then we have that E_N converges to a continuous operator E from H^δ to H^0. Since

$$L_\mu = \langle D_x\rangle^\mu L\langle D_x\rangle^{-\mu} = L + R_\mu$$

with R_μ of order zero, such an operator is continuous from $H^{\mu+\delta}$ to H^μ for every μ. $\square$

4. Propagation of singularities

We can use the fundamental solution (3.11) to investigate the propagation of the singularities in the Cauchy problem (1.1) with initial data $u_0, u_1 \in H^{-\infty}(\mathbf{R}^n)$, both in the case of a strictly hyperbolic operator P under assumptions either (1.5) or (1.6) and in the case of a weakly hyperbolic operator under assumptions (1.9), (1.11).

For a distribution v in $\mathbf{R}^n$, as usual, we denote by $WF(v)$ the wave front set of v, so a point $(x_0, \xi_0) \in \mathbf{R}^n \times (\mathbf{R}^n \setminus \{0\})$ does not belong to $WF(v)$ if and only if there are a micro-elliptic operator $Q(x, D_x)$ of order zero at (x_0, ξ_0) and functions $a(x), b(x) \in C_0^\infty(\mathbf{R}^n)$ such that $aQbv \in C_0^\infty(\mathbf{R}^n)$.

The projection of $WF(v)$ on $\mathbf{R}^n_x$ is the singular support ($\operatorname{singsupp} v$) of v.

For a closed subset K of $[0, T]$, a conic closed subset W of $\mathbf{R}^n \times (\mathbf{R}^n \setminus \{0\})$ and for $\nu \in \mathbf{Z}_+$, we denote by

$$\mathcal{C}^{(\nu)}(t, K, s)W$$

the smallest closed conic set containing all the points

$$(x, \xi) = \mathcal{C}^{(\mu)}(t, t_1, \ldots, t_\mu, s)(y, \eta), \quad t_1, \ldots, t_\mu \in K, \ \mu \leqslant \nu, \ (y, \eta) \in W,$$

that is, all the points at the time t of composed bicharacteristic curves that start from points in W at the time s and that may change bicharacteristic curve at the times $t_1, \ldots, t_\mu \in K$, $t \geqslant t_1 \geqslant \cdots \geqslant t_\mu \geqslant s$, $\mu \leqslant \nu$. If the number μ of points in K can be taken arbitrary large, then we use the notation

$$\mathcal{C}^{(\infty)}(t, K, s)W;$$

in the case $K = [s, t]$ we write

$$\mathcal{C}^{(\nu)}(t, s)W, \qquad \mathcal{C}^{(\infty)}(t, s)W.$$

From the action of Fourier integral operators on wave front sets, we have

$$WF\left(\int_s^t \int_s^{t_1} \cdots \int_s^{t_{\nu-1}} a_{\Psi^\nu}(t, t_1, \ldots, t_\nu, s; x, D_x)v \, dt_\nu \cdots dt_1\right) \subset \mathcal{C}^{(\nu)}(t, s)WF(v) \tag{4.1}$$

which gives a first rough estimate of the propagation of singularities by means of the fundamental solution $E(t, s)$ in (3.11):

$$WF\big(E(t, s)V\big) \subset \mathcal{C}^{(\infty)}(t, s)WF(V). \tag{4.2}$$

From this, one just deduces that the speed of propagation is finite. In fact, if we denote

$$\delta\big((x, \xi), (y, \eta)\big) = |x - y| + \big|\xi|\xi|^{-1} - \eta|\eta|^{-1}\big|,$$

then, (4.2) and (3.3) imply

$$\delta\big(WF\big(E(t, s)V\big), WF(V)\big) \leqslant c|t - s| \tag{4.3}$$

with

$$c = \sup_{j,t,x,|\xi|=1} \big|\nabla_\xi \lambda_j(t, x, \xi)\big| + \sup_{j,t,x,|\xi|=1} \big|\nabla_x \lambda_j(t, x, \xi)\big|.$$

Next, we improve (4.1) showing that the flux of singularities may change bicharacteristic curve only at times either in the singular support of the amplitude $a(t, t_1, \ldots, t_\nu, s; x, \eta)$ or corresponding to points where the roots λ_1 and λ_2 coincide. In particular, in the case of strictly hyperbolic operators with C^∞ coefficients in all variable, one re-obtains the well-known propagation given by the flux of simple bicharacteristics $\mathcal{C}_j(t, s)$, $j = 1, 2$.

Lemma 4.1. *For $s, t \in [0, T]$, $s < t$, let us consider the operator*

$$A_{\Psi^\nu} = A_{\Psi^\nu}(t, s; x, D_x) = \int_s^t \int_s^{t_1} \cdots \int_s^{t_{\nu-1}} a_{\Psi^\nu}(t, t_1, \ldots, t_\nu, s; x, D_x)v \, dt_\nu \cdots dt_1$$

with a regular amplitude in the variables $t_1, \dots, t_\nu$

$$a \in C^\infty\big([s,t]^\nu; S^m\big).$$

Suppose that

$$\big|\lambda_1(\tau,x,\xi) - \lambda_2(\tau,x,\xi)\big| \geqslant C_0\langle\xi\rangle, \quad C_0 > 0,\ \tau \in [s,t],$$

for all x *and all large* $|\xi|$. *Then, for every* $v \in H^{-\infty}(\mathbf{R}^n)$, *we have*

$$WF(A_{\Psi^\nu} v) \subset \big\{\mathcal{C}_j(t,s)(y,\eta);\ (y,\eta) \in WF(v),\ j = 1,2\big\}. \tag{4.4}$$

Proof. From (3.7) we have

$$\big|\partial_{t_j}\Psi^\nu(t,t_1,\dots,t_\nu,s,x,\xi)\big| \geqslant C_0\langle\xi\rangle, \quad C_0 > 0,$$

so we can integrate by parts using

$$e^{i\Psi^\nu} = \big(i\partial_{t_j}\Psi^\nu\big)^{-1}\partial_{t_j}\big(e^{i\Psi^\nu}\big), \quad j = 1,\dots,\nu.$$

After N integrations, $N \geqslant \nu$, taking also the second equality in (3.7) into account, we find

$$A_{\Psi^\nu}(t,s;x,D_x) = a^1_{m-\nu,\varphi_1}(t,s;x,D_x) + a^2_{m-\nu,\varphi_2}(t,s;x,D_x)$$
$$+ \sum_{k=0}^{\nu-1}\int_s^t\int_s^{t_1}\cdots\int_s^{t_k} a^{(k)}_{m-N,\Psi^{k+1}}(t,t_1,\dots,t_{k+1},s,x,D_x)\,dt_{k+1}\,dt_k\cdots dt_1,$$

with $a^j_{m-\nu} \in S^{m-\nu}$, $j = 1,2$, $a^{(k)}_{m-N} \in S^{m-N}$, $k = 0,\dots,\nu-1$, $t_0 = t$.

Thus, if we take $v \in H^\mu(\mathbf{R}^n)$ and $(x_0,\xi_0) \neq \mathcal{C}_j(t,s)(y,\eta)$, $j = 1,2$, for all $(y,\eta) \in WF(v)$ then we have $A_{\Psi^\nu}(t,s;x,D_x)v \in H^{\nu-m+N}$ micro-locally at (x_0,ξ_0) for any N which gives (4.4). □

We can now state the results about the propagation of singularities in problem (1.1).

Theorem 4.2. *Let* u *be the solution of the Cauchy problem for a strictly hyperbolic operator* P *in* (1.2) *satisfying either* (1.5) *or* (1.6) *and let us denote*

$$K = \operatorname{singsupp}\big(t \to Q_1(t,\cdot,\cdot)\big) \cup \operatorname{singsupp}\big(t \to Q_2(t,\cdot,\cdot)\big).$$

If the boundary ∂K *of* K *has zero Lebesgue measure, then the function* $u(t,x)$ *satisfies*

$$WF\big(u(t,\cdot)\big) \cup WF\big(\partial_t u(t,\cdot)\big) \subset \mathcal{C}^{(\infty)}(t,K,0)WF(u_0) \cup \mathcal{C}^{(\infty)}(t,K,0)WF(u_1). \tag{4.5}$$

Theorem 4.3. *Let us consider the Cauchy problem* (1.1) *for the operator* P *in* (1.9) *under the assumptions* (1.10), (1.11). *Let us take* T *such that the fundamental solution* (3.11) *constructed in Theorem* 3.1 *is defined for* $t,s \in [0,T]$ *and let* M *be the cardinality of the finite set*

$$K = \big\{t \in [0,T]:\ \alpha(t) = 0\big\}.$$

Then the solution $u(t,x)$ *satisfies*

$$WF\big(u(t,\cdot)\big) \cup WF\big(\partial_t u(t,\cdot)\big) \subset \mathcal{C}^{(M)}(t,K,0)WF(u_0) \cup \mathcal{C}^{(M)}(t,K,0)WF(u_1). \tag{4.6}$$

Proof of Theorems 4.2, 4.3. In both cases, it is sufficient to prove

$$WF\big(U(t,\cdot)\big) \subset \mathcal{C}^{(\infty)}(t,K,0)WF(U_0) \tag{4.7}$$

for the solution of (2.5), taking into account that

$$\mathcal{C}^{(\infty)}(t,K,0)WF(U_0) = \mathcal{C}^{(M)}(t,K,0)WF(U_0)$$

if K has finite cardinality M.

For any $\varepsilon > 0$, we can take a covering of the compact set ∂K (of the finite set K itself in Theorem 4.3) of total length less or equal to ε

$$\partial K \subset \bigcup_{j=1}^{M_\varepsilon} [a_j, b_j[, \qquad \sum_{j=1}^{M_\varepsilon} (b_j - a_j) \leqslant \varepsilon, \qquad [a_j, b_j[\cap [a_h, b_h[= \emptyset \quad \text{for } h \neq j. \tag{4.8}$$

We rename

$$0 = \tau_0 \leqslant \tau_1 \leqslant \tau_2 \leqslant \cdots \leqslant \tau_\mu \leqslant \tau_{\mu+1} = t$$

the points $a_j, b_j \in [0, t]$ and represent the solution U of (2.5) by means of the fundamental solution

$$U(t) = E(t, \tau_\mu) \cdots E(\tau_1, 0) U_0.$$

We apply Lemma 4.1 in the intervals $[\tau_j, \tau_{j+1}]$ such that $[\tau_j, \tau_{j+1}] \cap K = \emptyset$, the inclusion (4.2) when $[\tau_j, \tau_{j+1}] \subset K$ (this may happen only in Theorem 4.2), and the property (4.3) in the remaining intervals. We obtain

$$\delta\bigl(WF\bigl(U(t)\bigr), \mathcal{C}^{(\infty)}(t, K, 0) WF(U_0)\bigr) \leqslant c\varepsilon$$

so (4.7) letting $\varepsilon \to 0$. □

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